# LTB solutions in Newtonian gauge: from strong to weak fields.

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**Abstract**

Lemaître-Tolman-Bondi (LTB) solutions are used frequently to describe the collapse or expansion of spherically symmetric inhomogeneous mass distributions in the Universe. These exact solutions are obtained in the synchronous gauge where nonlinear dynamics (with respect to the FLRW background) induce large deviations from the FLRW metric. In this paper we show explicitly that this is a gauge artefact (for realistic sub-horizon inhomogeneities). We write down the nonlinear gauge transformation from synchronous to Newtonian gauge for a general LTB solution using the fact that the peculiar velocities are small. In the latter gauge we recover the solution in the form of a weakly perturbed FLRW metric that is assumed in standard cosmology. Furthermore we show how to obtain the LTB solutions directly in Newtonian gauge and illustrate how the Newtonian approximation remains valid in the nonlinear regime where cosmological perturbation theory breaks down. Finally we discuss the implications of our results for the backreaction scenario.

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# 1 Introduction

The Friedmann-Lemaître-Roberston-Walker (FLRW) framework is the cornerstone of modern cosmology. Its key assumption is that our large scale Universe is isotropic and homogeneous up to small perturbations. Until not so long ago this was really an assumption, but fortunately, present observations like the Sloan Digital Sky Survey show that for distance scales $\gtrsim 100$Mpc one does indeed seem to find statistical homogeneity [1][1]. At the same time however we know that at small scales $\lesssim 10$Mpc our Universe looks like anything but the idealized FLRW Universe. Nearly all of the matter has clumped into clusters and galaxies with local density contrasts and dynamics that are clearly beyond a linear description on a FLRW background. Given the nonlinear nature of GR, one may wonder then if the FLRW description of our large scale Universe is really justified. And if the corresponding scale factor evolves according to the Friedmann equations, with an effective matter density that is simply the averaged actual matter density.

This issue was studied in the past but gained much interest over the last few years with the discovery of the apparent cosmic acceleration (see [2, 3] for a review and an extensive list of references). Within the conventional FLRW framework this acceleration demands the presence of a cosmological constant, some other form of dark energy or a large distance modification of gravity. All cases require the ad hoc introduction of a new mass scale in the Lagrangian that is suspiciously of the same magnitude as the present Hubble constant. In light of this, several papers advocate the so called backreaction scenario [3, 4, 5, 6, 7, 8, 9]. According to this scenario the conventional FLRW framework is not justified. The hope is then that a correct averaging of the inhomogeneities at small distance scales would lead to an effective energy-momentum tensor to be used in the FLRW description at large scales, that would obviate the need for dark energy or modified gravity. A nice feature of this scenario is that it would naturally solve the coincidence problem (why now?) since it ties the epoch of cosmic acceleration to the epoch of nonlinear structure formation.

However, a strong argument against this scenario was raised by Ishibashi and Wald. In [10] they simply point out that, despite the nonlinear dynamics at small scales, the actual space-time metric of our Universe seems to be very well approximated by a

[1] That is if we don't consider the possibility of us living at the center of a Hubble-sized void.

Newtonianly perturbed FLRW metric at all scales. By this, they mean a metric of the form [2]

$$ds^2 = -\left(1+2\psi\right)dt^2 + a(t)^2\left(1-2\psi\right)\left(dx^2+dy^2+dz^2\right), \tag{1.1}$$

with

$$|\psi| \ll 1\ ,\ \ (\partial_t\psi)^2 \ll \frac{1}{a^2}\partial_i\psi\partial_i\psi\ ,\ \ (\partial_i\psi\partial_i\psi)^2 \ll (\partial_i\partial_j\psi)(\partial_i\partial_j\psi)\ ,\dots . \tag{1.2}$$

*If* (1.1) is really a good approximation everywhere to the metric of the Universe, then it is evident that the backreaction scenario does not work. Indeed, in that case one can immediately show that the scale factor $a(t)$ in the metric evolves according to the usual Friedmann equations, up to small corrections. Of course what we observe are rays of light, not a metric. But given a metric of the form (1.1) one can resort to a standard lensing analysis to show that the effect of the small scale inhomogeneities on cosmological observations is indeed small and can not explain away dark energy. (See [11] for instance, for an estimation of the effect in the context of supernovae observations.)

Now, one thing is the observation that the nonlinear dynamics at small scales do *not preclude* a weakly perturbed FLRW metric. Another, stronger statement, is that the metric (1.1) *actually is* a good approximation to the exact metric of the Universe. To argue the latter one has to resort to an expansion in the peculiar velocity $v$ of the full set of Einstein equations. In this expansion, that lies at the root of the Newtonian approximation in comoving coordinates, one finds that (1.1) is indeed the leading order approximation to the metric and that the higher order terms are small. However, this perturbative argument could in principle still be invalidated by non-perturbative effects.

In this paper we study this issue in the case of spherically symmetric space-times, the advantage being that we can use the exact LTB solutions [12] to check the validity of the Newtonian approximation. Not so surprising, the conclusion will be that the Newtonian approximation is valid both in the linear and nonlinear regime. As such this serves as a nice illustration of the compatibility of a global Newtonianly perturbed FLRW metric with local nonlinear dynamics. A subtlety will be that the LTB solutions are formulated in the synchronous gauge. In this gauge, starting from small initial

[2] For simplicity we are assuming flat FLRW here.

perturbations, the weak field description of the metric breaks down at the same time when the dynamics go nonlinear. In the next section we will show this explicitly for the LTB solutions, by linearizing the exact solutions on the FLRW background to recover the standard results of cosmological perturbation theory. In section 3, we will show however that this phenomenon is specific to the synchronous gauge. We will explicitly demonstrate that, for small peculiar velocities, the corresponding metric in Newtonian gauge remains of the weakly perturbed FLRW form throughout the entire evolution. In doing so, we will also verify that the solution that one obtains in the Newtonian gauge by solving the equations in the Newtonian approximation, agrees very well with the corresponding exact LTB solution.

The gauge transformations from synchronous to Newtonian gauge have been considered before in the context of LTB solutions. A valid application of the linear transformations was used in [13], an invalid application can be found in [14]. In [15] the full nonlinear transformation was performed on a specific LTB solution, obtaining a different result than ours. We will comment on this further on.

# 2 LTB solutions

The LTB solutions are exact solutions to the Einstein equations for spherically symmetric distributions of irrotational pressureless dust, so with energy-momentum tensor $T_{\mu\nu}(r,t) = \rho u_\mu u_\nu$. They are obtained in the synchronous gauge ($g_{00} = -1, g_{0i} = 0$) and from now on we will label the radial and time coordinate in this gauge as $(\tilde{r}, \tilde{t})$, reserving $(r, t)$ for the Newtonian gauge. Notice also that for pressureless dust, the synchronous gauge coincides with the comoving gauge ($u^\mu = (1, 0, 0, 0)$). Explicitly the solutions look like [3]:

$$ds^2 = -d\tilde{t}^2 + \frac{R'^2(\tilde{r},\tilde{t})}{1+2E(\tilde{r})}d\tilde{r}^2 + R(\tilde{r},\tilde{t})^2 d\Omega^2\,, \tag{2.3}$$

[3] We use units such that $c = 1$.

with[4]:

$$\frac{\dot{R}^2(\tilde{r},\tilde{t})}{2} + R(\tilde{r},\tilde{t})\ddot{R}(\tilde{r},\tilde{t}) = \frac{\dot{R}^2(\tilde{r},\tilde{t})}{2} - \frac{G_N M(\tilde{r})}{R(\tilde{r},\tilde{t})} = E(\tilde{r})\,, \tag{2.4}$$

$$4\pi\rho(\tilde{r},\tilde{t}) = \frac{M'(\tilde{r})}{R'(\tilde{r},\tilde{t})R^2(\tilde{r},\tilde{t})}\,, \tag{2.5}$$

solved by (with $E(\tilde{r}) \in \mathbb{R}$):

$$R = \frac{G_N M}{2E}\left(\cosh\left(u\sqrt{2E}\right) - 1\right)\,, \tag{2.6}$$

$$\tilde{t} - t_b(\tilde{r}) = \frac{G_N M}{2E}\left(\frac{\sinh(u\sqrt{2E})}{\sqrt{2E}} - u\right)\,. \tag{2.7}$$

A particular solution is then specified by some set of initial conditions $\{\rho(\tilde{r},\tilde{t}_i), R(\tilde{r},\tilde{t}_i), \dot{R}(\tilde{r},\tilde{t}_i)\}$ at time $\tilde{t}_i$ that translate to a certain choice for the free functions $\{E(\tilde{r}), M(\tilde{r}), t_b(\tilde{r})\}$ via eqs. (2.4)-(2.7). One of the initial conditions fixes the residual gauge freedom, we will choose $R(\tilde{r},\tilde{t}_i) = a_i\tilde{r}$ , where $a_i \equiv a(\tilde{t}_i)$ is the scale factor at time $\tilde{t}_i$ of the FLRW background at infinity. The two other initial conditions fix the initial density and *peculiar velocity* $v$, where the latter is defined as ($H \equiv \dot{a}/a$):

$$v \equiv \dot{R} - HR\,. \tag{2.8}$$

Some comments are in order before moving on. For the remainder of the paper we will only consider solutions that approach the Einstein- de Sitter solution (flat, matter dominated FLRW, with $a(t) = (t/t_0)^{2/3}$ ) at infinity, but it's straightforward to extrapolate our results to situations with non-flat FLRW backgrounds. One could even add radiation and/or dark energy for the FLRW background at infinity by taking slightly modified LTB solutions. Furthermore we will restrict ourselves to solutions that at some early time were close to the FLRW solution, in accord with the standard picture of the early Universe. It's also clear that $v(\tilde{r},\tilde{t})$ corresponds to the velocity of the shell of matter labelled by $\tilde{r}$ with respect to the FLRW background. Of course, by definition, the peculiar velocity in comoving/synchronous gauge is zero. But we will show later on that $v$ does indeed correspond to the proper peculiar velocity in Newtonian gauge (at least if $v \ll 1$).

[4]Some conventions for the derivatives: the prime stands for a radial and the dot for a time derivative with respect to the arguments of the function at hand. Later on we will also use the coordinates $(r,t)$ in Newtonian gauge, so we will have for instance $\alpha' = \alpha'(r,t) \equiv \frac{\partial\alpha}{\partial r}$ or $\partial_r R = \partial_r R(\tilde{r},\tilde{t}) = \dot{R}\partial_r\tilde{t} + R'\partial_r\tilde{r}$ .

Let us now consider some particular solution, describing the evolution of an initial inhomogeneity, characterized by a velocity and density profile at some early time $\tilde{t}_i$:

$$\begin{aligned} v_i(\tilde{r}) &\equiv v(\tilde{r}, \tilde{t}_i)\,, && (2.9)\\ \rho_i(\tilde{r}) &\equiv \rho(\tilde{r}, \tilde{t}_i) \equiv \rho_{FLRW}(\tilde{t}_i)(1+\delta(\tilde{r}, \tilde{t}_i)) \equiv \rho_{FLRW}(\tilde{t}_i)(1+\delta_i(\tilde{r}))\,, && (2.10)\end{aligned}$$

with

$$\delta_i, v_i \ll 1 \quad , \quad \lim_{\tilde{r}\to 0} v_i(\tilde{r}) = 0 \quad , \quad \lim_{\tilde{r}\to\infty} v_i(\tilde{r}) = 0 \quad , \quad \int_0^\infty d\tilde{r}\, \tilde{r}^2 \delta_i(\tilde{r}) = 0\,. \qquad (2.11)$$

From eqs. (2.4)-(2.7) and the first Friedmann equation we immediately find:

$$\begin{aligned} M(\tilde{r}) &= \frac{4\pi\tilde{r}^3 a_i^3}{3}\rho_{FLRW}(\tilde{t}_i)(1+3I_{\delta_i})\,, && (2.12)\\ E(\tilde{r}) &= v_i H_i a_i \tilde{r} - \frac{3}{2}(H_i a_i \tilde{r})^2 I_{\delta_i} + \frac{v_i^2}{2}\,, && (2.13)\\ t_b(\tilde{r}) &\approx \frac{1}{H_i}\left(\frac{2}{5}\frac{E}{(a_i H_i \tilde{r})^2} + I_{\delta_i}\right)\,, && (2.14)\end{aligned}$$

where we have defined

$$I_{\delta_i}(\tilde{r}) \equiv \frac{1}{\tilde{r}^3}\int_0^{\tilde{r}} dr'\, r'^2 \delta_i(r')\,. \qquad (2.15)$$

It is instructive now to expand the exact solution for $R(\tilde{r}, \tilde{t})$ in $\delta_i, v_i$ (treating both quantities as small parameters of the same order). This is precisely the expansion that is considered in cosmological perturbation theory to obtain an approximate solution to the equations of motion for more generic (non-symmetric) initial conditions. For the case at hand we find at leading order (keeping the full expression of $E \sim v_i \sim \delta_i$):

$$R(\tilde{r}, \tilde{t}) \approx \tilde{r} a(\tilde{t})\left(1 + \left(\left(\frac{\tilde{t}}{\tilde{t}_i}\right)^{2/3}\left(\frac{2}{5}\frac{E}{(a_i H_i \tilde{r})^2}\right) + I_{\delta_i} - \frac{\tilde{t}_i}{\tilde{t}}\left(\frac{2}{5}\frac{E}{(a_i H_i \tilde{r})^2} + I_{\delta_i}\right)\right) + \dots\right)\,, \qquad (2.16)$$

with the dominant part (for $\tilde{t} \gg \tilde{t}_i$) of the higher order terms going like

$$\sim \left(\left(\frac{\tilde{t}}{\tilde{t}_i}\right)^{2/3}\frac{E}{(a_i H_i \tilde{r})^2}\right)^n\,. \qquad (2.17)$$

We recognize the first term in the linear term corresponding to the growing mode, and the last term corresponding to the decaying mode of cosmological perturbation theory (see for instance [16] ). Furthermore, from the behavior of the higher order terms we see that the expansion breaks down at a time $t_{nl}$ when

$$\left(\frac{t_{nl}}{\tilde{t}_i}\right)^{2/3}\frac{E}{(a_i H_i \tilde{r})^2} \sim 1\,. \qquad (2.18)$$

Now, as we already commented in the introduction, one thing is the breakdown of the expansion in $\delta_i$, another thing is the breakdown of the metric description in terms of a weakly perturbed FLRW metric. But as one can see from (2.16), the breakdown of the former automatically implies the breakdown of the latter for the synchronous gauge solution that we are considering. Indeed we find

$$\frac{|R(\tilde{r},\tilde{t}) - a(\tilde{t})\tilde{r}|}{a(\tilde{t})\tilde{r}} \sim 1\,, \tag{2.19}$$

at $\tilde{t} \sim t_{nl}$. In addition one can see from the exact expression for the matter density (2.5), that the density contrast also becomes large at the same time: $\delta(\tilde{r}, t_{nl}) \sim 1$.

Notice that the converse is not necessarily true, one can have solutions with shell-crossing singularities ($g_{\tilde{r}\tilde{r}} = R' = 0$), for which the description of the metric in terms of a weakly perturbed FLRW clearly breaks down, that are perfectly well described by the perturbative expansion in $\delta_i$ for times arbitrary close to the time of shell-crossing. The "small u expansion" of [17] uses precisely this feature in combination with the exact expression (2.5) to describe the density.

# 3 From synchronous to Newtonian gauge (and back)

## 3.1 The linear case

We saw in the previous section how the description of the LTB metric as a weakly perturbed FLRW metric in synchronous gauge, breaks down at the same time when the dynamics on the FLRW background become nonlinear. Let us now see what happens in Newtonian (or Poisson) gauge. In spherically symmetric situations, given the standard angular coordinates, this gauge is specified by the conditions $g_{rt} = 0$ and $g_{rr}r^2 = g_{\theta\theta}$. So we want to find the coordinate transformation $(\tilde{r},\tilde{t}) \to (r,t)$ that changes the metric to:

$$ds^2 = -dt^2\left(1 + 2\phi(r,t)\right) + a^2(t)\left(dr^2 + r^2 d\Omega^2\right)\left(1 - 2\psi(r,t)\right)\,. \tag{3.20}$$

At times $t \ll t_{nl}$, when the dynamics are still linear, the coordinate transformation will also be linear[5] and one can resort directly to the machinery developed in the context of cosmological perturbation theory to obtain the potentials [16]. This was done for

[5] By a linear coordinate transformation we mean that $\tilde{t} = t + \alpha$ and $\tilde{r} = r + \beta$ with $\alpha', \dot{\alpha}, \beta', \dot{\beta} \ll 1$.

instance in [13] and for our conventions we find in a completely similar fashion that the resulting potentials read:

$$\phi \approx \psi + \mathcal{O}(\psi)^2 \approx -\frac{3}{5}\int\! dr \frac{E(r)}{r} \;+\; \frac{3}{2}\left(\frac{t_i}{t}\right)^{5/3} \int\! dr \left(\frac{2}{5}\frac{E(r)}{r} + r(a_i H_i)^2 I_{\delta_i}(r)\right) . \quad (3.21)$$

One can again recognize the growing mode and the decaying mode. As we should, we find the former to result in a constant term for the potential in Newtonian gauge. This, in contrast with the situation for synchronous gauge, implies that the metric keeps its form of a weakly perturbed FLRW ($\psi \approx \phi \ll 1$) up to the time $t_{nl}$, when the dynamics become nonlinear. Of course one can not resort to cosmological perturbation theory anymore to argue for the smallness of the fields at later times, when the fields and the actual density contrast $\delta$, become nonlinear functions of $\delta_i$. Still, from the argument of Ishibashi and Wald, we might expect the potential ($\psi \approx \phi$) to remain small for sub-horizon inhomogeneities. As we will show explicitly in the next subsection, this is indeed the case for the LTB solutions in Newtonian gauge, provided that the peculiar velocities remain small. For the LTB solutions this will hold as long as we stop the evolution well before a central singularity develops. In reality, when we consider the collapse of a cluster for instance, the (effective) pressure that arises during the virialization, will halt the collapse.

## 3.2 The nonlinear case

We could proceed now in the same way as we did for the linear case. So, starting from the LTB solution we could look for the coordinate transformation $(\tilde{r}, \tilde{t}) \rightarrow (r, t)$, which is now nonlinear, to obtain the metric in the Newtonian gauge. This strategy was used in [15]. But we find it more instructive go the other way. We will start from the equations in Newtonian gauge for spherically symmetric situations, with metric (3.20), showing explicitly how the expansion in the peculiar velocity $v$ gives rise to the familiar equations of the Newtonian approximation to GR. Then we will perform the coordinate transformation to synchronous gauge, demonstrating explicitly that the solutions obtained in the Newtonian approximation are indeed good approximations to the exact LTB solutions.

First we should explain at last what we mean exactly by the expansion in $v$. From Newtonian physics one gets the following order of magnitude estimates in the case of

a non-virialized system with density contrast $\delta$ and distance scale $L$[18]:

$$\psi \sim \phi \sim \delta (HL)^2 \ , \ \ \psi' \sim \delta H^2 L \ , \ \ v \sim \delta HL \ , \ \ \dot{\psi} \sim \psi \frac{v}{L} \ , \ \ \dot{v} \sim \frac{v^2}{L} \ , \ldots . \tag{3.22}$$

When assigning a power of $v$ to a particular term in the expansion, we will use the order of magnitude estimates above, with $\delta \sim 1$. This ensures that the expansion remains valid in the nonlinear regime $\delta \gtrsim 1$.

Let us now apply this expansion to the full set of Einstein equations. In the spherically symmetric case that we are considering, there are four independent equations:

$$G_{tt} \approx 8\pi G_N \rho \ , \ G_{tr} \approx -8\pi G_N \rho v a \ , \ G_{rr} \approx 8\pi G_N \rho v^2 a^2 \ , \ G_{\varphi\varphi} = \sin^2\theta \, G_{\theta\theta} = 0 \, , \tag{3.23}$$

for the perfect dust energy-momentum tensor $T_{\mu\nu} = \rho u_\mu u_\nu$, with $u^\mu \approx (1, v/a, 0, 0)$. We easily get the fields $\phi$ and $\psi$ from the $(\theta,\theta)$ and $(t,t)$ equations. The former

$$G_{\theta\theta} = r^2(\phi'' - \psi'' + \frac{1}{r}\phi' - \frac{1}{r}\psi') \ + \mathcal{O}(v^4) = 0 \, , \tag{3.24}$$

immediately tells us that $\psi = \phi + \mathcal{O}(v^4)$.[6] The latter then reduces to the Poisson equation (using the FLRW equation for the background), with the potential sourced by the density contrast:

$$\nabla^2 \phi = 4\pi G_N a^2 \rho_{FLRW} \delta \ + H^2 \mathcal{O}(v^2) \, . \tag{3.25}$$

The metric (3.20) will then solve the other two Einstein equations if the energy-momentum tensor is conserved. That is if $\rho$ and $v$ obey the familiar ideal fluid equations in comoving coordinates (for zero pressure) [18]:

$$\nabla_\mu T^{\mu t} = \left( \dot{\rho} \ + 3H\rho + \frac{1}{a}(\rho v)' + \frac{2}{ra}\rho v \right) + \frac{H^2}{G_N L}\mathcal{O}(v^3) \ = \ 0 \, , \tag{3.26}$$

$$\frac{a}{\rho}\left( \nabla_\mu T^{\mu r} - \frac{v}{a}\nabla_\mu T^{\mu t} \right) = \left( \dot{v} + \frac{v v'}{a} + H v + \frac{\phi'}{a} \right) + \frac{1}{L}\mathcal{O}(v^4) \ = \ 0 \, . \tag{3.27}$$

The solution in the Newtonian approximation can now be obtained by solving the eqs. (3.25)-(3.27) at leading order, for a particular initial velocity and density profile $v_i(r), \delta_i(r) \ll 1$ at some early time $t_i$. One can verify that for these solutions, the order of magnitude estimates (3.22) are correct, so the higher order terms in the

[6] We are imposing the boundary conditions $\phi, \psi, \phi', \psi' \to 0$ for $r \to \infty$, to match the FLRW background at infinity.

expansion will indeed be suppressed, both in the linear ($\delta \ll 1$) and nonlinear ($\delta \gtrsim 1$) regime as long as $v$ remains small. This type of argument on the validity of the Newtonian approximation could be criticized for being circular. In a sense we are using the Newtonian approximation to justify itself. But as we will now show by going to the synchronous gauge, the approximate solution obtained in the Newtonian gauge, is indeed a very good approximation to the exact LTB solution.

We define the coordinate transformation by:

$$
\begin{aligned}
R(\tilde{r},\tilde{t})^2 &= a(t)^2 r^2 (1-2\psi(r,t))\,, && (3.28)\\
\tilde{t} &= t+\alpha(r,t)\,, && (3.29)
\end{aligned}
$$

for some functions $R(\tilde{r},\tilde{t})$ and $\alpha(r,t)$ that will be determined from the synchronous gauge conditions $g_{\tilde{t}\tilde{r}} = 0$ and $g_{\tilde{t}\tilde{t}} = -1$. We will do this in the same expansion in $v$ that we used to obtain the Newtonian approximation, retaining only the terms up to $\mathcal{O}(v^2)$. Immediately we can anticipate that

$$
\alpha' \approx -av\,, \qquad (3.30)
$$

from the condition that $u^{\tilde{r}}$ should be zero in the synchronous/comoving gauge. Applying the coordinate transformation on the metric, with the use of

$$
\begin{aligned}
d\tilde{t} &= dt\,(1+\dot{\alpha}) + dr\alpha'\,, && (3.31)\\
R'd\tilde{r} &\approx dt\left(\dot{a}r - \dot{R}\right) + dr\left(a(1-\psi) - ar\psi' - \dot{R}\alpha'\right)\,, && (3.32)
\end{aligned}
$$

we then find

$$
g_{\tilde{t}\tilde{r}} \approx \frac{\alpha' R'}{a} + R'(\dot{R} - \dot{a}r) = 0\,, \qquad (3.33)
$$

for

$$
\dot{R} - \dot{a}r \approx \dot{R} - HR \approx -\frac{\alpha'}{a}\ (\approx v)\,. \qquad (3.34)
$$

Employing this expression it is then straightforward to show that $g_{\tilde{t}\tilde{t}} \approx -1$, if $v$ obeys the Euler equation (3.27). So the transformation (3.28)-(3.29) with $\alpha$ obeying (3.30) and $R$ obeying (3.34) indeed takes us to the synchronous gauge.

We will now show that $R$ approximately solves the LTB equation if $v, \rho$ and $\phi$ solve the eqs. (3.25)-(3.27), that we obtained in the Newtonian approximation. Let us first look at the continuity equation (3.26) and show that it is equivalent to the LTB expression (2.5) for the matter density. Expressing the $(r,t)$ derivatives in terms of

$(\tilde{r},\tilde{t})$ derivatives through eqs. (3.31) and (3.32), this equation becomes (again keeping the appropriate powers of $v$):

$$\partial_{\tilde{t}}\rho + 3H\rho + \rho\frac{\partial_{\tilde{r}} v}{R'} + \frac{2}{ra}\rho v \approx 0\,. \tag{3.35}$$

Using the expressions (3.28) and (3.34) for $R$ and $v$, this reduces to

$$\partial_{\tilde{t}}\rho + \rho\left(\frac{\dot{R}'}{R'} + 2\frac{\dot{R}}{R}\right) \approx 0\,, \tag{3.36}$$

solved by the LTB expression (2.5) for $\rho$.

Employing this LTB expression for $\rho$ in the Poisson equation (3.25) for $\phi$, we find in a similar way that:

$$\begin{aligned}
\phi' &\approx \frac{a^2}{r^2}\left(\int dr\, r^2\left(\frac{G_N M'(\tilde{r})}{R'R^2} - \frac{3}{2}H^2\right)\right)\\
&\approx \frac{a}{R^2}\left(\int d\tilde{r}\, G_N M'(\tilde{r})\right) - \frac{aH^2R}{2}\\
&\approx a\left(\frac{G_N M}{R^2} - \frac{H^2R}{2}\right)\,.
\end{aligned} \tag{3.37}$$

If we now use this expression for $\phi'$ in combination with $a(t) = (t/t_0)^{2/3}$, we finally find that the Euler equation (3.27) reduces to:

$$\ddot{R} + \frac{G_N M(\tilde{r})}{R^2} = \frac{1}{\dot{R}}\frac{\partial}{\partial\tilde{t}}\left(\frac{\dot{R}^2}{2} - \frac{G_N M(\tilde{r})}{R}\right) \approx 0, \tag{3.38}$$

which is now indeed solved by the LTB equation (2.4). So we have demonstrated that for a solution obtained in the Newtonian approximation one finds $g_{\theta\theta} = R(\tilde{r},\tilde{t})^2$ in the synchronous gauge, with $R$ an approximate solution of the exact LTB equations. Keeping track of the omitted terms in our expansion one can show that the approximation holds up to terms $\sim v^4$. As for the initial conditions, it is easy to see that the initial density and velocity profile at time $t_i$ in the Newtonian gauge, translate to (approximately) the same initial conditions at time $\tilde{t}_i$ for the LTB solutions: $\delta_i(r), v_i(r) \approx \delta_i(\tilde{r}), v_i(\tilde{r})$. That is if $r \approx \tilde{r}$ around the time $t_i$, which is true if we fix the residual gauge degree of freedom by the condition $R(\tilde{r},\tilde{t}_i) = a_i\tilde{r}$, as we did in the previous section.

A nice cross check of our derivation is provided by the calculation of $g_{\tilde{r}\tilde{r}}$. From the exact LTB solution we know that we should find

$$g_{\tilde{r}\tilde{r}} = \frac{R'^2}{1+2E}\,, \tag{3.39}$$

whereas from the coordinate transformation (3.31)-(3.32) we find,

$$g_{\tilde{r}\tilde{r}} \approx R'^2 \left(1 + 2\left(\frac{\dot{a}}{a}\alpha' r + \psi' r\right) - \frac{\alpha'^2}{a^2}\right) . \tag{3.40}$$

Using the expressions for $R$ (3.28) and $\alpha$ (3.34) this indeed reduces to

$$\begin{aligned} g_{\tilde{r}\tilde{r}} &\approx R'^2 \left(1 - \dot{R}^2 + \frac{2G_N M}{R}\right) \\ &\approx \frac{R'^2}{1+2E} , \end{aligned} \tag{3.41}$$

where on the last line we have used the LTB equation (2.4) and $E \ll 1$.

To recapitulate, we have demonstrated that the coordinate transformation, implicitly defined by:

$$R(\tilde{r},\tilde{t})^2 = a(t)^2 r^2 (1 - 2\psi(r,t)) , \tag{3.42}$$

$$\tilde{t} = t + a(t)\int_r^\infty dr' v(r',t) , \tag{3.43}$$

with

$$\dot{R}(\tilde{r},\tilde{t}) - H(\tilde{t})R(\tilde{r},\tilde{t}) = v(r,t) , \tag{3.44}$$

takes the metric (3.20) in Newtonian gauge to synchronous gauge, provided $v$ is small. In the synchronous gauge we recover the metric of the LTB form (2.3), with $R$ and $\rho$ approximately obeying the LTB equations (2.4) and (2.5) in synchronous coordinates, if $\rho, v, \phi \approx \psi$ are solutions of the Poisson, continuity and Euler equations in the Newtonian coordinates.

It's clear that we can use our results also in the other direction. Namely if we start from the exact LTB solution in synchronous gauge, the transformation (3.42)-(3.43) will take the metric to a Newtonianly perturbed FLRW metric with $\rho, \psi, v$ solving the Poisson, continuity and Euler equation in Newtonian coordinates at leading order in $v$. We should stress (at the risk of being repetitive) that our analysis is valid both in the linear and nonlinear regime, as long as the peculiar velocities remain small. Notice also that the conditions on $\psi$ (1.2) imposed by Ishibashi and Wald are automatically satisfied, again if $v$ is small.

As we mentioned, in [15] a similar analysis was performed for the specific case of zero initial velocity and an initial density profile describing a constant over-density

in the core surrounded by a finite region with a constant under-density. Surprisingly, the authors found a considerable difference for the fields $\psi$ and $\phi$ when the dynamics go nonlinear, seemingly in conflict with the Newtonian approximation and with our results. However, it turns out that the reason for the discrepancy lies in a calculational error [19].

# 4 Conclusions

In this paper we have demonstrated explicitly how one can recover the exact LTB solutions in synchronous gauge, from the corresponding solutions obtained in the Newtonian approximation. This was done by applying the full nonlinear coordinate transformation from Newtonian to synchronous gauge, using the fact that the peculiar velocities remain small, which is the case for realistic sub-horizon inhomogeneities.

As we have illustrated, in their original form the LTB solutions display a breakdown of the weak field description of the metric, when the dynamics go nonlinear with respect to the FLRW background. This has been used to argue for the backreaction scenario by Rasanen [3, 4] for instance. Indeed, once the weak field description breaks down, one would not a expect a priori that a Universe full of collapsing structures and expanding voids would still give a FLRW metric *on the average* with a normal evolution of the scale factor. However, as we have shown, the breakdown of the weak field description of the metric is in fact specific to the synchronous gauge. In Newtonian gauge, the LTB solutions are perfectly well described by a Newtonianly perturbed FLRW metric and it becomes straightforward to show that both the metric and the dynamics will on the average behave as a conventional FLRW Universe (up to small corrections).

LTB solutions have been used also in the literature to model the effect of inhomogeneities on light propagation in the so called Swiss cheese models [20]. Again, given the breakdown of the weak field description, one would not expect a priori to approximately recover the standard FLRW luminosity-distance redshift relation, when averaging over all directions. Yet this is precisely what is found and is again easy to understand from the weak field expansion in Newtonian gauge, as was demonstrated for LTB solutions in the linear regime in [13]. In a forthcoming publication we will use the full transformation (3.42)-(3.43) to analyze the Swiss cheese model of [21] in the nonlinear regime. We will find that the metric indeed reduces to a weakly perturbed

FLRW metric in the Newtonian gauge, which was used implicitly in the recent paper [22].

All this illustrates the strength of the Newtonian approximation *in comoving coordinates*[7] in justifying the conventional FLRW framework. And it is clear that any serious backreaction scenario should explain where and how the Newtonian approximation breaks down. Actually we know that the Newtonian approximation breaks down in the vicinity of black holes, and to our knowledge it has not been fully demonstrated yet that the corresponding backreaction is negligible. Notice also that in this paper we have only strictly proven the validity of the Newtonian approximation for spherically symmetric space-times, that approach the FLRW solution at infinity. But as we commented in the introduction, for more general space-times one can show that the Newtonian approximation is at least self consistent.

Finally we should comment on the local void scenario, which is another approach that involves the use of inhomogeneity in trying to dispose of dark energy. In this scenario one puts us near the center of a large void, typically described by an LTB solution. The mismatch between the local and global expansion can then explain the supernovae data [23, 24, 25]. However, to explain other data sets, like those on the CMB and the large scale structure, one has to introduce additional features in the primordial power spectrum and the matter composition of the Universe [26, 27]. Another contrast with the backreaction scenario is that this scenario can be perfectly well described in a weak field description, with the local Hubble flow encoded in the Doppler term (see for instance [13]). Indeed, a good fit to the supernovae data requires a shift in the local Hubble parameter of order 10% with respect to the global value that is recovered at a distance $r \sim 1\text{Gpc}$. This translates to a maximal peculiar velocity $v \sim \Delta H\, r \lesssim 0.05$ which is still rather small and therefore validates the Newtonian approximation. This issue was studied by the authors of [14], for the type of void proposed in [24]. For this void, the present peculiar velocity is indeed small, and since the solution is in the nonlinear regime, one needs the full nonlinear transformation (3.42)-(3.43) to recover the metric as a Newtonian perturbation of the FLRW metric. Also, as was noticed in [14], a specific feature of this void solution is that the peculiar velocities become large ($v \gtrsim 1$) in the past, at redshifts $z \gtrsim 500$. At that time the void represents a nonlinear

[7] As opposed to the Newtonian approximation in physical coordinates, which breaks down at cosmological distance scales.

super-horizon density fluctuation and the weak field description in Newtonian gauge will of course break down.


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